

# QCD Sum Rules for Heavy-Meson Decay Constants: Impact of Renormalization Scale and Scheme

Wolfgang Lucha\*, Dmitri Melikhov\*,<sup>†</sup> and Silvano Simula\*\*

\**Institute for High Energy Physics, Austrian Academy of Sciences, Nikolsdorfergasse 18, A-1050 Vienna, Austria*

<sup>†</sup>*D. V. Skobeltsyn Institute of Nuclear Physics, M. V. Lomonosov Moscow State University, 119991, Moscow, Russia*

\*\**INFN, Sezione di Roma Tre, Via della Vasca Navale 84, I-00146, Roma, Italy*

**Abstract.** Within the realm of QCD sum rules, one of the most important areas of application of this nonperturbative approach is the prediction of the decay constants of heavy mesons. However, in spite of the fact that, indisputably, the adopted techniques are, of course, very similar, we encounter rather dissimilar challenges, or obstacles, when extracting from two-point correlators of appropriate heavy-light currents interpolating the mesons, the characteristics of charmed mesons with different spin. In view of this, it seems worthwhile to us to revisit this issue for the case of charmed pseudoscalar mesons  $D_{(s)}$  and vector mesons  $D_{(s)}^*$ .

**Keywords:** quantum chromodynamics, QCD sum rules, SVZ sum rules, charmed mesons, heavy-meson decays, decay constant, pseudoscalar meson, vector meson, operator product expansion, Borel transformation, quark–hadron duality, renormalization scheme, renormalization scale

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## APPROACHING BOUND STATES OF STRONG INTERACTIONS BY QCD SUM RULES

Any description of physical systems bound by the strong interactions that deserves to be attributed as *reliable* should be based on quantum chromodynamics (QCD, the quantum field theory that governs the strong interactions) and should be of *non-perturbative* nature. One formalism that — in contrast to, for instance, lattice gauge theory — offers the prospect of providing analytical insights, namely, in form of relations between features of hadrons and the parameters of QCD is realized by the technique of *QCD sum rules* [1]. Their formulation proceeds along a well-established sequence of steps:

- Define the correlation function of a nonlocal product of operators (in particular, of appropriate quark currents) that interpolate the hadron under study, i.e., have nonvanishing matrix elements between vacuum and this hadron state.
- Evaluate this correlation function at the hadron level, by inserting a complete set of states, and at the QCD level, by applying Wilson’s operator product expansion (OPE) reshaping any nonlocal product to a series of local operators, to obtain perturbative contributions, represented by dispersion integrals of spectral densities, and non-perturbative (NP) terms, labelled as “power” contributions, representing the “vacuum condensates” of the local OPE operators.
- Get rid of subtraction terms left behind by Cauchy’s integral formula and suppress the effects of hadron excitations and continuum, by performing a Borel transformation from momentum to another variable, the Borel parameter  $\tau$ .
- Hide your ignorance about higher states by postulating quark–hadron duality; thus assume that all contributions of hadronic excited and continuum states cancel against those of perturbative QCD above effective thresholds  $s_{\text{eff}}(\tau)$ .

## DECAY CONSTANTS OF PSEUDOSCALAR AND VECTOR CHARMED MESONS $D_{(s)}^{(*)}$

Taking advantage of the experimental knowledge [2] of the masses  $M_{P,V}$  of the mesons discussed, our goal is to perform advanced [3–12] extractions of the decay constants,  $f_{P,V}$ , of both pseudoscalar (P) [13, 14] and vector (V) [15] charmed mesons (regarded as bound states of a charmed quark  $c$  of mass  $m_c$  and, in the non-strange case, of a light quark  $q = d$  of mass  $m_d$  or, in the strange case, of a light quark  $q = s$  of mass  $m_s$ ) from the two-point correlation functions of adequately chosen interpolating currents. As indicated, by way of construction the QCD sum rules derived along the lines sketched above are expressed, at QCD level, in terms of spectral densities  $\rho^{(P,V)}(s, \mu)$  and non-perturbative terms  $\Pi_{\text{NP}}^{(P,V)}(\tau, \mu)$  at appropriate renormalization scale  $\mu$ . Terming the QCD side of such sum rule as the *dual* correlator,  $\tilde{\Pi}_{P,V}(\tau, s_{\text{eff}}(\tau))$ , we refer to the characteristics predicted by this sum rule for a ground-state meson as its *dual* mass and *dual* decay constant:

- For the charmed pseudoscalar mesons  $P = D, D_s$ , we select as their interpolating operator the pseudoscalar current  $j_5(x) \equiv (m_c + m_q) \bar{q}(x) i \gamma_5 c(x)$  to extract [13, 14] both  $M_P$  and decay constants  $f_P$ , defined by  $\langle 0 | j_5(0) | P \rangle = f_P M_P^2$ :

$$f_P^2 M_P^4 \exp(-M_P^2 \tau) = \int_{(m_c+m_q)^2}^{s_{\text{eff}}(\tau)} ds \exp(-s \tau) \rho^{(P)}(s, \mu) + \Pi_{\text{NP}}^{(P)}(\tau, \mu) \equiv \tilde{\Pi}_P(\tau, s_{\text{eff}}(\tau)) ,$$

$$M_{\text{dual}}^2(\tau) \equiv -\frac{d}{d\tau} \log \tilde{\Pi}_P(\tau, s_{\text{eff}}(\tau)) , \quad f_{\text{dual}}^2(\tau) \equiv \frac{\exp(M_P^2 \tau)}{M_P^4} \tilde{\Pi}_P(\tau, s_{\text{eff}}(\tau)) .$$

- For the charmed vector mesons  $V = D^*, D_s^*$ , we use as interpolating operator the vector current  $j_\mu(x) \equiv \bar{q}(x) \gamma_\mu c(x)$  to obtain [15] the masses  $M_V$  and decay constants  $f_V$ , defined by  $\langle 0 | j_\mu(0) | V(p) \rangle = f_V M_V \epsilon_\mu(p)$  from the sum rule

$$f_V^2 M_V^2 \exp(-M_V^2 \tau) = \int_{(m_c+m_q)^2}^{s_{\text{eff}}(\tau)} ds \exp(-s \tau) \rho^{(V)}(s, \mu) + \Pi_{\text{NP}}^{(V)}(\tau, \mu) \equiv \tilde{\Pi}_V(\tau, s_{\text{eff}}(\tau)) ,$$

$$M_{\text{dual}}^2(\tau) \equiv -\frac{d}{d\tau} \log \tilde{\Pi}_V(\tau, s_{\text{eff}}(\tau)) , \quad f_{\text{dual}}^2(\tau) \equiv \frac{\exp(M_V^2 \tau)}{M_V^2} \tilde{\Pi}_V(\tau, s_{\text{eff}}(\tau)) .$$

The spectral densities are known to three-loop accuracy [16, 17]; the values of our OPE parameters are listed in Table 1.

**TABLE 1.** Input values (in  $\overline{\text{MS}}$  renormalization scheme) chosen for quark masses, QCD coupling and the lowest-dimensional vacuum condensates.

OPE parameter	Numerical input value
$\bar{m}_d(2 \text{ GeV})$	$(3.42 \pm 0.09) \text{ MeV}$
$\bar{m}_s(2 \text{ GeV})$	$(93.8 \pm 2.4) \text{ MeV}$
$\bar{m}_c(\bar{m}_c)$	$(1275 \pm 25) \text{ MeV}$
$\alpha_s(M_Z)$	$0.1184 \pm 0.0020$
$\langle \bar{q}q \rangle(2 \text{ GeV})$	$-[(267 \pm 17) \text{ MeV}]^3$
$\langle \bar{s}s \rangle(2 \text{ GeV})$	$(0.8 \pm 0.3) \times \langle \bar{q}q \rangle(2 \text{ GeV})$
$\left\langle \frac{\alpha_s}{\pi} GG \right\rangle$	$(0.024 \pm 0.012) \text{ GeV}^4$

## PROGRESSING TOWARDS IMPROVED PREDICTIONS OF HADRON OBSERVABLES

For Borelized QCD sum rules, progress in the achieved precision [3–7] may be hampered by too conventional attitudes:

1. The requirement of Borel stability is nothing but a reflection of one’s mere hope that the value of a hadronic feature predicted by a QCD sum rule at an extremum in the Borel parameter is a reliable approximation to the actual value, but may lead one astray, as experience with the counterparts of such sum rules in quantum mechanics shows [3–7].
2. The probably very naïve but persistently defended belief that the effective threshold does not know about the Borel parameter [8–12], i.e., the assumption that the effective threshold is constant, is just a result of not knowing better.<sup>1</sup>

In view of this, we proposed to allow for the easy-to-find Borel parameter dependence of the effective threshold [8–12]:

- Determine the range of admissible Borel parameters  $\tau$  — the “working Borel window” — by the requirement that, at the window’s lower end, the contribution of the ground state is sufficiently large and, at the window’s upper end, the contributions of the nonperturbative corrections are still reasonably small. For our analysis, this yields [13–15]

$$0.1 \text{ GeV}^{-2} < \tau < 0.5 \text{ GeV}^{-2} \quad \text{for } D, D^*, D_s^* , \quad 0.1 \text{ GeV}^{-2} < \tau < 0.6 \text{ GeV}^{-2} \quad \text{for } D_s .$$

<sup>1</sup> Apart from our enduring campaign [8–12] against such oversimplifying point of view, a notable exception is an investigation of the decay constants of heavy–light mesons reported in Ref. [18], which hiddenly makes use of an implicit dependence of the continuum threshold on the Borel parameter.

- To derive the Borel-parameter dependence of the effective thresholds  $s_{\text{eff}}(\tau)$ , adopt the simple polynomial Ansatz<sup>2</sup>

$$s_{\text{eff}}^{(n)}(\tau) = \sum_{j=0}^n s_j \tau^j, \quad n = 0, 1, 2, \dots,$$

and pin down its coefficients  $s_j$  by minimizing over a set of  $N$  equidistant discrete points  $\tau_i$  in the Borel window the squared difference of dual meson mass squared  $M_{\text{dual}}^2(\tau_i)$  and experimentally measured meson mass squared  $M_{\text{P,V}}^2$

$$\chi^2 \equiv \frac{1}{N} \sum_{i=1}^N [M_{\text{dual}}^2(\tau_i) - M_{\text{P,V}}^2]^2, \quad N = 1, 2, 3, \dots$$

- Having played around with several toy sum rules in quantum mechanics [3–12], feel entitled to interpret the spread of results for the polynomial degree  $n = 1, 2, 3$  as a hint to the size of the *intrinsic* error of a QCD sum-rule finding.

## SYSTEMATIC UNCERTAINTIES FROM RENORMALIZATION SCHEME AND SCALE

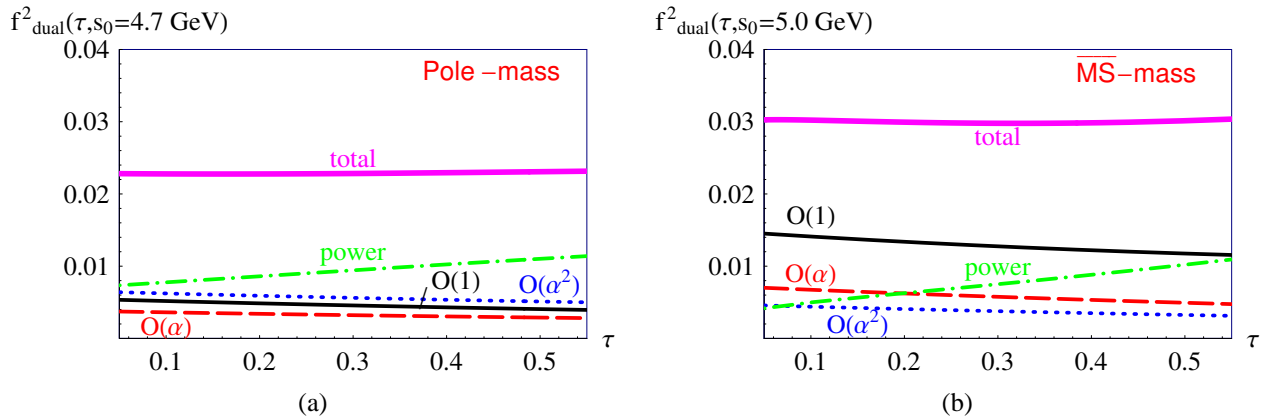
### Issue: optimization of the perturbative convergence of OPE contributions to QCD sum rules

Perturbation theory enables us to derive the coefficient multiplying a given local operator in some OPE in the form of a series in powers of the strong coupling,  $\alpha_s(\mu)$ . The one of the unit operator ends up in the perturbative spectral density

$$\rho(s, m_c, \mu) = \rho_0(s, m_c) + \frac{\alpha_s(\mu)}{\pi} \rho_1(s, m_c) + \frac{\alpha_s^2(\mu)}{\pi^2} \rho_2(s, m_c, \mu) + \dots$$

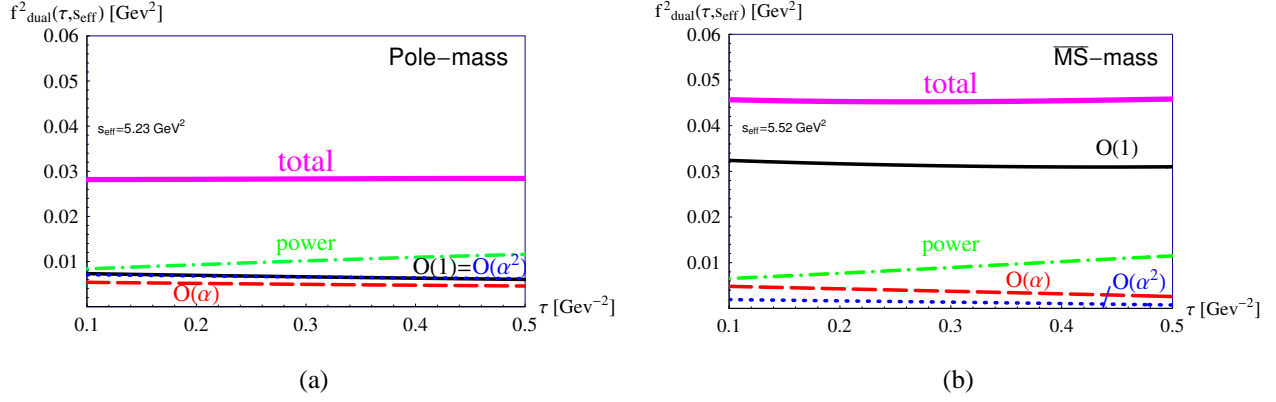
For the relative importance of the contributions both of different order in  $\alpha_s$  and of power corrections to predicted decay constants, the choice of the renormalization scheme defining the  $c$ -quark mass,  $m_c$ , makes a big difference: although the central values are compatible within errors, the comparisons shown, for the  $D$  meson, in Fig. 1 and, for the  $D^*$  meson, in Fig. 2 assign a greater credibility to results deriving from use of the  $\overline{\text{MS}}$  running mass  $m_c = \overline{m}_c(\overline{m}_c) = (1275 \pm 25) \text{ MeV}$  than to those relying on the pole mass  $m_c = \hat{m}_c = 1699 \text{ MeV}$ , related to the former, via known expressions  $r_1, r_2$  [19], by

$$\overline{m}_c(\mu) = \hat{m}_c \left( 1 + \frac{\alpha_s(\mu)}{\pi} r_1 + \frac{\alpha_s^2(\mu)}{\pi^2} r_2 + \dots \right).$$



**FIGURE 1.** Breakdown of the OPE contributions to the dual decay constant  $f_D(\tau)$  of the nonstrange charmed pseudoscalar meson  $D$  [13, 14], extracted for a *fixed* threshold  $s_0$  in either pole-mass renormalization scheme (a) or  $\overline{\text{MS}}$ -mass renormalization scheme (b).

<sup>2</sup> Note that this Ansatz allows for or covers (for  $n = 0$ ) but also generalizes the conventional prejudice that the effective threshold should be constant.



**FIGURE 2.** Disentanglement of the OPE contributions to the dual decay constant  $f_{D^*}(\tau)$  of the nonstrange charmed *vector* meson  $D^*$  [15] predicted, for a *fixed* threshold  $s_{\text{eff}}$ , by either pole-mass renormalization scheme (a) or  $\overline{\text{MS}}$ -mass renormalization scheme (b).

### Issue: dependence of QCD sum-rule extractions of decay constants on renormalization scale

Needless to recall, *physical observables* do not care about intermediate technicalities such as renormalization scales: they do not depend on any renormalization scales. Ideally, also theoretical descriptions of such quantities should not do. Unfortunately, within the formalism of QCD sum rules, for practical reasons inevitable truncations, in their perturbative contributions, to merely *finite order* of the expansions in powers of the coupling parameter and, in their nonperturbative power corrections, to the relevant vacuum condensates of *lowest dimensions* induce an artificial unphysical dependence on renormalization scales. In the case of decay constants, upon introducing the average  $\overline{\mu}$  of the renormalization scale  $\mu$  by defining  $f_{\text{dual}}(\overline{\mu}) = \langle f_{\text{dual}}(\mu) \rangle$ , such findings are approximately reproduced by power series in the logarithm of  $\mu/\overline{\mu}$ :

$$f_{D(s)}^{(*)}(\mu) = a \left( 1 + c_1 \log \frac{\mu}{\overline{\mu}} + c_2 \log^2 \frac{\mu}{\overline{\mu}} + c_3 \log^3 \frac{\mu}{\overline{\mu}} \right). \quad (1)$$

The numerical values of the scale average  $\overline{\mu}$  and of the parameters  $a, c_1, c_2, c_3$  entering in this expansion, emerging from our QCD sum-rule extraction of the decay constants of the charmed mesons  $D, D_s, D^*$ , and  $D_s^*$ , are collected in Table 2. The average scale  $\overline{\mu}$  is somewhat larger for vector mesons than for pseudoscalar mesons. As evident from the numerical values of primarily the coefficient  $c_1$ , the sensitivity of the charmed-meson decay constants to the renormalization scale  $\mu$ , depicted in Fig. 3, is definitely more pronounced for charmed vector mesons than for charmed pseudoscalar mesons.

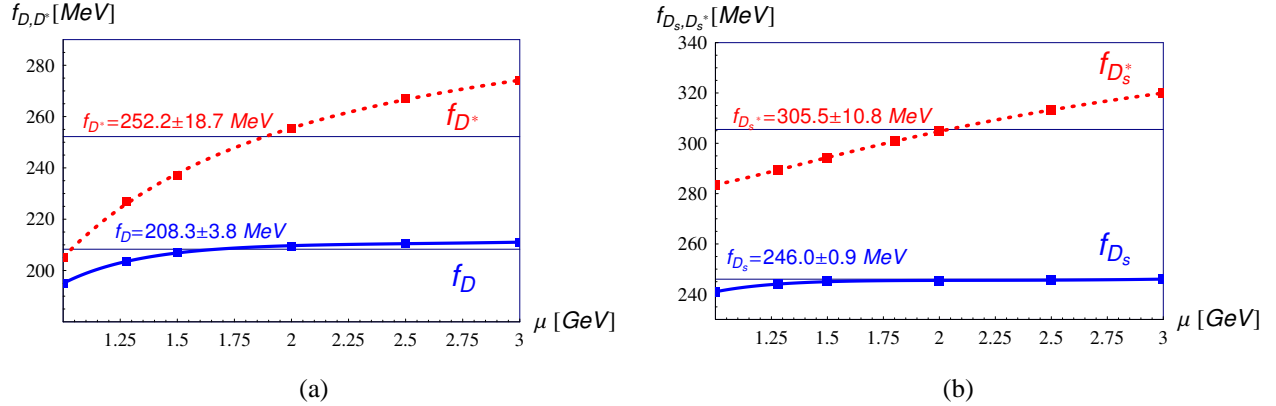
**TABLE 2.** Outcomes [15] for average scale  $\overline{\mu}$  and coefficients  $a, c_1, c_2, c_3$  in the parametrization (1) of the decay-constant renormalization scale behaviour.

Charmed meson	$\overline{\mu}$ (GeV)	$a$ (MeV)	$c_1$	$c_2$	$c_3$
$D$	1.62	208.3	+0.06	-0.11	+0.08
$D_s$	1.52	246.0	+0.01	-0.03	+0.04
$D^*$	1.84	252.2	+0.233	-0.096	+0.17
$D_s^*$	1.94	305.5	+0.124	+0.014	-0.034

### DECAY CONSTANTS OF THE CHARMED MESONS: QCD SUM-RULE PREDICTIONS

In this analysis, our goal was to take a fresh look, from a common perspective, at our separate extractions [13–15] of the decay constants of both pseudoscalar and vector charmed mesons. Qualitatively, we arrive at the following conclusions:

- When it comes to the hierarchy of the perturbative and nonperturbative OPE contributions, use of the  $c$ -quark mass defined by the  $\overline{\text{MS}}$  renormalization scheme is the clear favourite of both pseudoscalar and vector charmed mesons.
- *Central values* of decay constants derived using the  $\overline{\text{MS}}$   $c$  mass are 30% larger than those found from its pole mass.



**FIGURE 3.** QCD sum-rule predictions for the unphysical dependence of the dual decay constants  $f_{D^{(*)}}$  of the charmed nonstrange mesons  $D, D^*$  (a) and of the dual decay constants  $f_{D_s^{(*)}}$  of the charmed strange mesons  $D_s, D_s^*$  (b) on the renormalization scale  $\mu$  [15].

- Unlike pseudoscalar mesons, vector mesons (viz., the decay-constant errors) take notice of renormalization scales.

Quantitatively, our predictions for the decay constants of both pseudoscalar [13, 14] and vector [15] charmed mesons (including the *OPE-related* errors, caused by the uncertainties of the parameter values entering as input to the OPE, and the *systematic* errors, due to the inherently limited accuracy of the QCD sum-rule approach) are summarized in Table 3.

**TABLE 3.** Reevaluation of the decay constants of the charmed *pseudoscalar* mesons  $D_{(s)}$  [13, 14] and *vector* mesons  $D_{(s)}^*$  [15] by our improved algorithm.

Charmed meson	Decay constant $f_{D_{(s)}^{(*)}}$ (MeV)
$D$	$206.2 \pm 7.3_{\text{OPE}} \pm 5.1_{\text{syst}}$
$D_s$	$245.3 \pm 15.7_{\text{OPE}} \pm 4.5_{\text{syst}}$
$D^*$	$252.2 \pm 22.3_{\text{OPE}} \pm 4_{\text{syst}}$
$D_s^*$	$305.5 \pm 26.8_{\text{OPE}} \pm 5_{\text{syst}}$

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